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Torsion of closed section, orthotropic, thin-walled beams

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Abstract

The paper presents a theory for thin-walled, closed section, orthotropic beams which takes into account the shear deformation in restrained warping induced torque. In the derivation we developed the analytical ("exact") solution of simply supported beams subjected to a sinusoidal load. The replacement stiffnesses which are independent of the length of the beam were determined from the exact solution by taking its Taylor series expansion with respect to the inverse of the length of the beam. The effect of restrained warping and shear deformation was investigated through numerical examples.

Keywords: Composite; Closed section; Beam theory; Torsion; Restrained warping

1. Introduction

Fiber reinforced plastic (composite), thin-walled beams are widely used in the aerospace industry and are increasingly applied in the infrastructure. Thin-walled beams are often made with closed cross-sections because of their high torsional stiffness.

Classical beam theories, which neglect bending–torsion coupling, transverse shear deformation and torsional warping stiffness often fail to predict the behavior of closed section, composite beams. To avoid the undesirable bending–torsion coupling, beams can be manufactured such that their layup is orthotropic (Kollár and Springer, 2003), (however not necessarily symmetrical).

In this paper a new theory is presented for orthotropic, closed section thin-walled beams taking transverse shear and restrained warping into account. There are composite beam theories (Massa and Barbero,

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1998; Rehfield et al., 1988) which take transverse shear and restrained warping into account, however they neglect the effect of shear deformation on restrained warping which may overestimate the warping stiffness. This effect is explained for pure torsion below:

Classical beam theories, derived by Vlasov (1961) and also included in classical textbooks (Megson, 1990), calculates the bimoment ($\widehat{M}_\omega$) and the Saint Venant torque ($\widehat{T}_{sv}$) as

$$\widehat{M}_\omega = \widehat{EI}_\omega \Gamma \qquad \widehat{T}_{sv} = \widehat{GI}_t \vartheta \tag{1}$$

where $\widehat{EI}_\omega$ is the warping stiffness, $\widehat{GI}_t$ is the torsional stiffness, ϑ is the rate of twist (which is the first derivative of the rotation of the cross-section $\vartheta = d\psi/dx$), and

$$\Gamma = -\frac{d\vartheta}{dx} \tag{2}$$

where x is the axial coordinate. The torque ($\widehat{T}$) is the sum of the Saint Venant torque ($\widehat{T}_{sv}$) and the restrained warping induced torque ($\widehat{T}_\omega$)

$$\widehat{T} = \widehat{T}_{sv} + \widehat{T}_\omega \tag{3}$$

where the latter is calculated as

$$\widehat{T}_\omega = -\frac{d\widehat{M}_\omega}{dx} \tag{4}$$

Eqs. (1)–(4) give the well-known equation:

$$\widehat{T} = \widehat{GI}_t \vartheta - \widehat{EI}_\omega \frac{d^2\vartheta}{dx^2} \tag{5}$$

In the theory, presented in this paper, we assume that the rate of twist (ϑ) consists of two parts

$$\vartheta = \vartheta_B + \vartheta_S \tag{6}$$

where subscripts "B" and "S" refer to the bending and shear deformations. (Note the similarity with the Thimoshenko beam theory for the inplane deformations of beams, where the first derivative of the displacement consists of two parts: $dv/dx = \chi + \gamma$, where the first term is the rotation of the cross-section and the second is the transverse shear strain.) $\widehat{T}_\omega$ is calculated from ϑ_S as

$$\widehat{T}_\omega = S_{\omega\omega} \vartheta_S \tag{7}$$

where $S_{\omega\omega}$ is the rotational shear stiffness. Eqs. (1), (3) and (4) are valid, however Eq. (2) is replaced by

$$\Gamma = -\frac{d\vartheta_B}{dx} \tag{8}$$

A theory, where the effect of shear deformation on restrained warping is taken into account (and the basic idea of which for pure torsion is explained above) was derived in Kollár (2001) for open section composite beams. This paper can be considered as the generalization of Kollár (2001) for closed section beams. Note that Roberts and Al-Ubaidi (2001) and Wu and Sun (1992) also proposed the use of Eq. (6). The paper of Roberts and Ubaidi only shows the importance of the effect but do not provide a complete theory, Wu and Sun's solution is rather complex and too tedious for design purposes.

The shear deformation in restrained warping may have a significant effect on short beams, and this effect is not included in Massa and Barbero (1998) and Rehfield et al. (1988) which is indicated by the empty boxes in the fifth column of Table 1.

Table 1
Comparison of composite beam theories

Beam models	Not isotropic	Not orthotropic	Restrained warping	Shear in warping	Arbitrary closed cross-section
Massa and Barbero (1998)	*				*
Mansfield and Sobey (1979)	*	Inaccurate for unsym. laminate			*
Rehfield et al. (1988)	*	Inaccurate for unsym. laminate	*		*
Kollár and Pluzsik (2002)	*	*			*
Urban (1955)			*	Inaccurate	Doubly sym. cross-section
Present	*		*	*	*

For thin-walled beams with symmetrical layup the effect of local bending stiffness is negligible, however for unsymmetrical layups it may have a significant effect, which was shown in Pluzsik and Kollár (2002), and hence we included the effect of local stiffness in the presented theory.

We must give credit to the work of Urban (1955), who developed a theory for closed section, *isotropic* beams with uniform cross-section. Urban took into account the shear deformation in restrained warping, however assumed a uniform shear flow which is not a reasonable assumption when the effect of restrained warping is significant. His theory was extended to non-uniform cross-sections (non-prismatic beams) by Kristek (1979). Both Urban and Kristek restricted their analysis for doubly symmetrical isotropic beams.

Vlasov—the pioneer of thin-walled beam theories—also presented a solution for isotropic, closed section beams containing of flat walls (Vlasov, 1961). In his solution, in pure torsion, he assumed independent warping functions for each wall-segment and hence no cross-sectional properties were presented, and hence, his solution is rather complex.

Below we summarize the governing equations of Kollár (2001) which was developed for open section composite beams. These equations will be generalized in this paper for closed section beams.

We consider transversely loaded, *open section*, orthotropic beams consisting of an arbitrary number of flat wall segments (Fig. 1). The twist has two parts: one from bending (which causes warping) and an other part from the restrained warping induced shear stress, as indicated by Eq. (6).

1.1. Basic assumptions

(1) The material of the cross-section behaves in a linearly elastic manner.
(2) The effect of the displacements of the axis of the beam is not taken into account in the equilibrium equations.
(3) The effect of change in geometry of the cross-section is not taken into account in the equilibrium equations.
(4) The Kirchhoff–Love hypotesis is valid for each plate element.

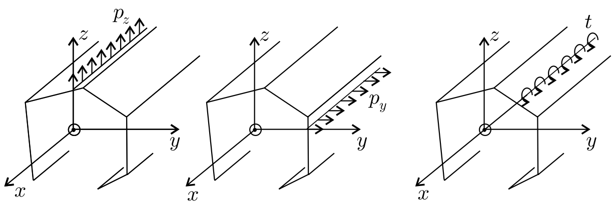

Fig. 1. Loads on a thin-walled beam.

(5) The normal stresses in the contour directions are small compared to the axial stresses.
(6) The form of the axial strain is

$$\epsilon_x^o = \frac{\mathrm{d}u}{\mathrm{d}x} - y\frac{\mathrm{d}\chi_y}{\mathrm{d}x} - z\frac{\mathrm{d}\chi_z}{\mathrm{d}x} - \omega\frac{\mathrm{d}\vartheta_\mathrm{B}}{\mathrm{d}x} \tag{9}$$

where u is the axial displacement, χ_y and χ_z are the rotations of the cross-section in the $x-y$ and $x-z$ planes, ϑ_B is the rate of twist from bending, and $\omega = \int_0^s r\,\mathrm{d}s$ is a section property called the sectorial area. The last term in Eq. (9) represents an additional axial displacement of the cross-section, called warping, proportional to the rate of twist from bending (Megson, 1990). χ_y, χ_z and ϑ_B can be calculated as follows:

$$\chi_y = \frac{\mathrm{d}v}{\mathrm{d}x} - \gamma_y \qquad \chi_z = \frac{\mathrm{d}w}{\mathrm{d}x} - \gamma_z \qquad \vartheta_\mathrm{B} = \frac{\mathrm{d}\psi}{\mathrm{d}x} - \vartheta_\mathrm{S} \tag{10}$$

where γ_y and γ_z are the shear strains and v and w are the displacements in the $x-y$ and $x-z$ planes, respectively, ψ is the twist and ϑ_S is the rate of twist from shear.

The shear strain is supposed to be constant in the cross-section which is referred to as the first order shear theory. Couplings between normal and shearing effects are neglected.

1.2. Governing equations

We summarize below the governing equations of open section, orthotropic thin-walled beams (Kollár, 2001), and present the expressions for calculating the shear stiffnesses.

The equilibrium equations in matrix form are as follows:

$$\begin{bmatrix} & & & & -\frac{\mathrm{d}}{\mathrm{d}x} & & \\ & & & & & -\frac{\mathrm{d}}{\mathrm{d}x} & \\ & & & -\frac{\mathrm{d}}{\mathrm{d}x} & & & -\frac{\mathrm{d}}{\mathrm{d}x} \\ \frac{\mathrm{d}}{\mathrm{d}x} & & & & -1 & & \\ & \frac{\mathrm{d}}{\mathrm{d}x} & & & & -1 & \\ & & \frac{\mathrm{d}}{\mathrm{d}x} & & & & -1 \end{bmatrix} \begin{Bmatrix} \widehat{M}_y \\ \widehat{M}_z \\ \widehat{M}_\omega \\ \widehat{T}_\mathrm{sv} \\ \widehat{V}_y \\ \widehat{V}_z \\ \widehat{T}_\omega \end{Bmatrix} = \begin{Bmatrix} p_y \\ p_z \\ t \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{11}$$

where p_y and p_z are the external loads in the $y-x$ and $z-x$ planes and t is the distributed torque (see Fig. 1). The internal shear forces $\widehat{V}_y$, $\widehat{V}_z$ are defined as

$$\widehat{V}_y = \int (N_{\xi\eta}\cos\alpha)\mathrm{d}s \qquad \widehat{V}_z = \int (N_{\xi\eta}\sin\alpha)\mathrm{d}s \tag{12}$$

The internal moments $\widehat{M}_y$, $\widehat{M}_z$, and $\widehat{M}_\omega$ are

$$\widehat{M}_y = \int (N_\xi y + M_\xi\cos\alpha)\mathrm{d}s \qquad \widehat{M}_z = \int (N_\xi z + M_\xi\sin\alpha)\mathrm{d}s \tag{13}$$

$$\widehat{M}_\omega = \int (N_\xi\omega)\mathrm{d}s \tag{14}$$

$N_{\xi\eta}$, N_{ξ}, M_{ξ} are the shear force, normal force and bending moment over unit length of the wall (Eq. (23)). The torque consists of two parts: the Saint Venant torque and the warping induced torque:

$$\widehat{T} = \widehat{T}_{\mathrm{sv}} + \widehat{T}_{\omega} \tag{15}$$

The stress–strain relationship is the following:

$$\begin{Bmatrix} \widehat{M}_y \\ \widehat{M}_z \\ \widehat{M}_\omega \\ \widehat{T}_{\mathrm{sv}} \\ \widehat{V}_y \\ \widehat{V}_z \\ \widehat{T}_\omega \end{Bmatrix} = \begin{bmatrix} \widehat{EI}_{yy} & \widehat{EI}_{yz} & & & & & \\ \widehat{EI}_{yz} & \widehat{EI}_{zz} & & & & & \\ & & \widehat{EI}_\omega & & & & \\ & & & \widehat{GI}_{\mathrm{t}} & & & \\ & & & & S_{yy} & S_{yz} & S_{y\omega} \\ & & & & S_{yz} & S_{zz} & S_{z\omega} \\ & & & & S_{y\omega} & S_{z\omega} & S_{\omega\omega} \end{bmatrix} \begin{Bmatrix} \frac{1}{\rho_y} \\ \frac{1}{\rho_z} \\ \Gamma \\ \vartheta \\ \gamma_y \\ \gamma_z \\ \vartheta_{\mathrm{s}} \end{Bmatrix} \tag{16}$$

where the generalized strains $\frac{1}{\rho_y}$, $\frac{1}{\rho_z}$ and Γ are

$$\frac{1}{\rho_y} = -\frac{\mathrm{d}\chi_y}{\mathrm{d}x} \qquad \frac{1}{\rho_z} = -\frac{\mathrm{d}\chi_z}{\mathrm{d}x} \qquad \Gamma = -\frac{\mathrm{d}\vartheta_{\mathrm{B}}}{\mathrm{d}x} \tag{17}$$

In the stiffness matrix $\widehat{EI}_{yy}$, $\widehat{EI}_{zz}$, $\widehat{EI}_{yz}$ are the bending stiffnesses, $\widehat{EI}_\omega$ is the warping stiffness, $\widehat{GI}_{\mathrm{t}}$ is the torsional stiffness and S_{ij} are the shear stiffnesses.

The strain–displacement relationship is given by

$$\begin{Bmatrix} \frac{1}{\rho_y} \\ \frac{1}{\rho_z} \\ \Gamma \\ \vartheta \\ \gamma_y \\ \gamma_z \\ \vartheta_{\mathrm{s}} \end{Bmatrix} = \begin{bmatrix} & & & -\frac{\mathrm{d}}{\mathrm{d}x} & & \\ & & & & -\frac{\mathrm{d}}{\mathrm{d}x} & \\ & & & & & -\frac{\mathrm{d}}{\mathrm{d}x} \\ & & \frac{\mathrm{d}}{\mathrm{d}x} & & & \\ \frac{\mathrm{d}}{\mathrm{d}x} & & & -1 & & \\ & \frac{\mathrm{d}}{\mathrm{d}x} & & & -1 & \\ & & \frac{\mathrm{d}}{\mathrm{d}x} & & & -1 \end{bmatrix} \begin{Bmatrix} v \\ w \\ \psi \\ \chi_y \\ \chi_z \\ \vartheta_{\mathrm{B}} \end{Bmatrix} \tag{18}$$

It can be seen that the shear deformation in torsion (ϑ_{s}) is defined analogously to the shear deformation in bending (γ_y and γ_z). We can calculate the bending, torsional and warping stiffnesses in the same way as for beams made of isotropic material (Massa and Barbero, 1998). Below we will give the calculation of the shear compliances which are defined as

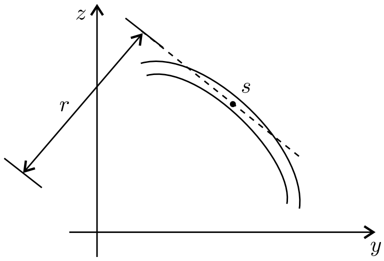

Fig. 2. Definition of r.

$$\begin{bmatrix} s_{yy} & s_{yz} & s_{y\omega} \\ s_{yz} & s_{zz} & s_{z\omega} \\ s_{y\omega} & s_{z\omega} & s_{\omega\omega} \end{bmatrix} = \begin{bmatrix} S_{yy} & S_{yz} & S_{y\omega} \\ S_{yz} & S_{zz} & S_{z\omega} \\ S_{y\omega} & S_{z\omega} & S_{\omega\omega} \end{bmatrix}^{-1} \tag{19}$$

According to Kollár (2001) the shear flow consists of three parts

$$q = \widehat{V}_y q_y + \widehat{V}_z q_z + \widehat{T}_\omega q_\omega \tag{20}$$

where q_y, q_z and q_ω are the shear flows caused by unit shear loads ($\widehat{V}_y = 1,\ \widehat{V}_z = 1$) and by a unit torque ($\widehat{T} = \widehat{T}_\omega = 1$), respectively. The shear flows q_y, q_z and q_ω can be calculated according to the classical analysis of thin-walled beams (Megson, 1990). The expressions of s_{yy}, s_{zz}, $s_{\omega\omega}$, s_{yz}, $s_{y\omega}$ and $s_{z\omega}$ are as follows (Kollár, 2001):

$$\begin{aligned} s_{yy} &= \int \alpha_{66} q_y^2 \,\mathrm{d}s \qquad s_{zz} = \int \alpha_{66} q_z^2 \,\mathrm{d}s \qquad s_{\omega\omega} = \int \alpha_{66} q_\omega^2 \,\mathrm{d}s \\ s_{yz} &= \int \alpha_{66} q_y q_z \,\mathrm{d}s \qquad s_{y\omega} = \int \alpha_{66} q_y q_\omega \,\mathrm{d}s \qquad s_{z\omega} = \int \alpha_{66} q_y q_\omega \,\mathrm{d}s \end{aligned} \tag{21}$$

where α_{66} is the shear compliance of the wall (see Eq. (23)) (Fig. 2).

2. Problem statement

We consider thin-walled closed section prismatic beams. The beam consists of flat segments (Fig. 3) designated by the subscript k ($k = 1, 2, \ldots, K$, where K is the total number of the wall segments). The cross-section may be symmetrical or unsymmetrical and the layup of the wall is orthotropic. The beam may be subjected to distributed loads (shown in Fig. 1) or to concentrated loads. We wish to determine the displacements of the beam.

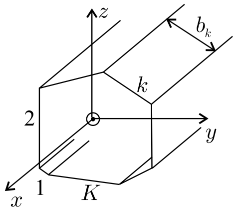

Fig. 3. Cross-section of the closed section, thin-walled beam.

3. Governing equations

We apply the first five assumptions given in Section 1. The sixth assumption will be used only in Section 5. We employ the following coordinate systems (Fig. 4).

For the beam we use the x–y–z coordinate system with the origin at the centroid. For the kth segment we employ the ξ_k–η_k–ζ_k coordinate system with the origin at the center of the reference plane of the kth segment. ξ is parallel to the x coordinate, η is along the circumference of the wall, and ζ is perpendicular to the circumference.

The axial displacements of an arbitrary point, s of the cross-section (Fig. 5) is given by Kollár and Springer (2003)

$$u(s) = -\int_0^s r\,\mathrm{d}\eta\,\vartheta + \int_0^s \gamma^0_{\xi\eta}\,\mathrm{d}\eta \tag{22}$$

where η is the circumferential coordinate, r is given in Fig. 2, ϑ is the rate of twist and $\gamma^0_{\xi\eta}$ is the shear strain.

For an orthotropic wall the stress–strain relationship is given as (Kollár and Springer, 2003)

$$\begin{Bmatrix} \epsilon^o_\xi \\ \epsilon^o_\eta \\ \gamma^0_{\xi\eta} \\ \kappa_\xi \\ \kappa_\eta \\ \kappa_{\xi\eta} \end{Bmatrix}_k = \begin{bmatrix} \alpha_{11} & \alpha_{12} & 0 & \beta_{11} & \beta_{12} & 0 \\ \alpha_{12} & \alpha_{22} & 0 & \beta_{21} & \beta_{22} & 0 \\ 0 & 0 & \alpha_{66} & 0 & 0 & \beta_{66} \\ \beta_{11} & \beta_{21} & 0 & \delta_{11} & \delta_{12} & 0 \\ \beta_{12} & \beta_{22} & 0 & \delta_{12} & \delta_{22} & 0 \\ 0 & 0 & \beta_{66} & 0 & 0 & \delta_{66} \end{bmatrix}_k \begin{Bmatrix} N_\xi \\ N_\eta \\ N_{\xi\eta} \\ M_\xi \\ M_\eta \\ M_{\xi\eta} \end{Bmatrix}_k \tag{23}$$

where the calculation of the elements of the compliance matrix (α_{ij}, β_{ij}, δ_{ij}) are given by Kollár and Springer (2003), $\epsilon^o_\xi, \epsilon^o_\eta, \gamma^0_{\xi\eta}$ are the strains of the reference surface of the wall, κ_ξ, κ_η, $\kappa_{\xi\eta}$ are the curvatures of the wall, N_ξ, N_η, $N_{\xi\eta}$ are the in-plane forces (per unit length) and M_ξ, M_η, $M_{\xi\eta}$ are the moments (per unit length) as illustrated in Fig. 6.

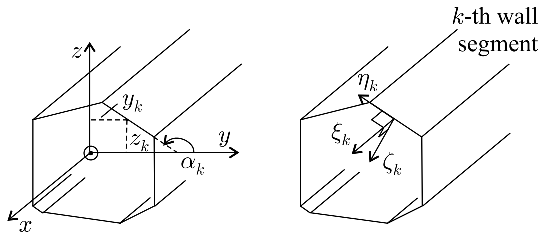

Fig. 4. Coordinate systems employed in the analysis of thin-walled beams with arbitrary layup.

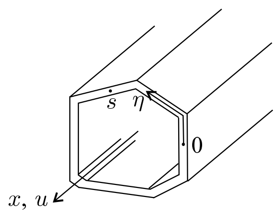

Fig. 5. Definition of u, s, η.

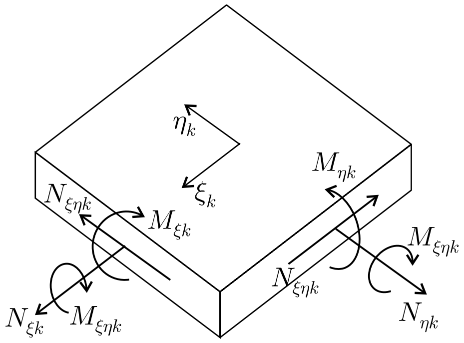

Fig. 6. In-plane forces and moments of a plate element.

The first and third rows of Eq. (23) are

$$\epsilon^o_{\xi k} = (\alpha_{11})_k N_{\xi k} + (\alpha_{12})_k N_{\eta k} + (\beta_{11})_k M_{\xi k} + (\beta_{12})_k M_{\eta k} \tag{24}$$

$$\gamma^0_{\xi\eta k} = (\alpha_{66})_k N_{\xi\eta k} + (\beta_{66})_k M_{\xi\eta k} \tag{25}$$

By definition $N_{\xi\eta k}$ is the shear flow, and we write

$$N_{\xi\eta k} = q \tag{26}$$

$N_{\eta k}$ and $M_{\eta k}$ are small and can be neglected (see Assumption 3)

$$N_{\eta k} \cong 0 \qquad M_{\eta k} \cong 0 \tag{27}$$

From Eqs. (24) and (25) we obtain

$$\epsilon^o_{\xi k} = (\alpha_{11})_k N_{\xi k} + (\beta_{11})_k M_{\xi k} \tag{28}$$

$$\gamma^0_{\xi\eta k} = (\alpha_{66})_k q + (\beta_{66})_k M_{\xi\eta k} \tag{29}$$

When the wall is symmetrical $(\beta_{ij})_k = 0$, and consequently Eqs. (28) and (29) become

$$N_{\xi k} = \frac{1}{(\alpha_{11})_k} \epsilon^o_{\xi k} \tag{30}$$

$$\gamma^0_{\xi\eta k} = (\alpha_{66})_k q \tag{31}$$

(Note however, that these relationships can be applied for unsymmetrical layups, provided that $(\alpha_{11})_k$ is evaluated at the ‘‘tension neutral’’ and $(\alpha_{66})_k$ at the ‘‘torque neutral’’ surface, see Appendix A of Pluzsik and Kollár (2002).)

By substituting Eqs. (22) and (31) (together with $\epsilon^o_{\xi k} = \mathrm{d}u/\mathrm{d}x$) into Eq. (30) we have

$$N_{\xi k}(s) = \frac{1}{(\alpha_{11})_k} \frac{\mathrm{d}}{\mathrm{d}x} \left(-\int_0^s r\, \mathrm{d}\eta\, \vartheta + \int_0^s \alpha_{66} q\, \mathrm{d}\eta \right) \tag{32}$$

The equilibrium equation in the axial direction (see Fig. 7) results in

$$\frac{\partial N_{\xi k}}{\partial x} + \frac{\partial q_k}{\partial \eta} = 0 \tag{33}$$

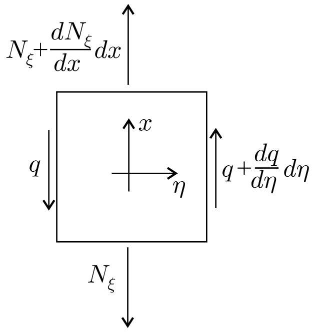

Fig. 7. Forces in the x direction on an element of the wall.

We substitute Eq. (32) into Eq. (33), and write

$$\frac{1}{(\alpha_{11})_k}\frac{\partial^2}{\partial x^2}\left(-\int_0^s r\,\mathrm{d}\eta\,\vartheta + \int_0^s \alpha_{66} q\,\mathrm{d}\eta\right) + \frac{\partial q_k}{\partial \eta} = 0 \tag{34}$$

By differentiating with respect to η, after algebraic manipulation, we obtain

$$-r_k\frac{\partial^2 \vartheta}{\partial x^2} + (\alpha_{66})_k\frac{\partial^2 q_k}{\partial x^2} + (\alpha_{11})_k\frac{\partial^2 q_k}{\partial \eta^2} = 0 \tag{35}$$

This second order differential equation is valid for every wall segment ($k = 1,\ldots,K$). The following continuity conditions must be satisfied.

The shear flow must be continuous, hence, we have

$$q_k|_{\frac{b_k}{2}} = q_{k+1}|_{-\frac{b_{k+1}}{2}} \quad k = 1,\ldots,K \tag{36}$$

The axial displacements (u) of the adjacent walls must be identical. A necessary condition is that the derivative of the axial strains are identical. Consequently, we write

$$(\alpha_{11})_k\frac{\partial q_k}{\partial \eta}\bigg|_{\frac{b_k}{2}} = (\alpha_{11})_{k+1}\frac{\partial q_{k+1}}{\partial \eta}\bigg|_{-\frac{b_{k+1}}{2}} \quad k = 1,\ldots,K \tag{37}$$

(Note that in the above equations $K + 1$ must be replaced by 1, see Fig. 3.)

4. Solution of the governing equations in pure torsion

We consider a simply supported beam (Fig. 8) subjected to a sinusoidal torque $t = \tilde{t}\sin \pi x/l$. At a simple support $\psi = 0$, $\psi'' = 0$. We assume that the beam undergoes pure torsion. (Pure torsion occurs either when

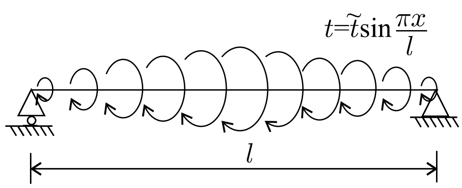

Fig. 8. Simply supported beam subjected to a sinusoidal torque.

the cross-section of the beam is doubly symmetrical or when the horizontal and vertical displacements (v, w) of the beam's axis are constrained.)

4.1. "Exact" solution of torsion for sinusoidal loads

The solution of the problem is assumed to be in the form of the following functions:

$$\vartheta = \widetilde{\vartheta}\cos\frac{\pi x}{l} \qquad q_k = \widetilde{q}_k(\eta)\cos\frac{\pi x}{l} \tag{38}$$

where $\widetilde{\vartheta}$ is a constant and $\widetilde{q}_k$ is a function of η only.

Note that these functions satisfy the boundary conditions at $x = 0$ and $x = l$. By substituting Eq. (38) into Eq. (35) we obtain

$$\left(r_k\frac{\pi^2}{l^2}\widetilde{\vartheta} - (\alpha_{66})_k\frac{\pi^2}{l^2}\widetilde{q}_k + (\alpha_{11})_k\frac{\partial\widetilde{q}_k}{\partial\eta^2}\right)\cos\frac{\pi x}{l} = 0 \tag{39}$$

which results in the following second order, ordinary, inhomogeneous differential equation:

$$(\alpha_{66})_k\frac{\pi^2}{l^2}\widetilde{q}_k - (\alpha_{11})_k\frac{\partial\widetilde{q}_k}{\partial\eta^2} = r_k\frac{\pi^2}{l^2}\widetilde{\vartheta} \tag{40}$$

The general solution is (Kreyszig, 1993)

$$\frac{\widetilde{q}_k}{\widetilde{\vartheta}} = \frac{r_k}{(\alpha_{66})_k} + C_{1,k}\mathrm{e}^{-\lambda_k\left(\frac{b_k}{2}+\eta\right)} + C_{2,k}\mathrm{e}^{-\lambda_k\left(\frac{b_k}{2}-\eta\right)} \tag{41}$$

where

$$\lambda_k = \frac{\pi}{l}\sqrt{\frac{(\alpha_{66})_k}{(\alpha_{11})_k}} \tag{42}$$

By substituting Eq. (41) into Eqs. (36) and (37) we have

$$C_{1,k}\mathrm{e}^{-\lambda_k b_k} + C_{2,k} - C_{1,k+1} - C_{2,k+1}\mathrm{e}^{-b_{k+1}\lambda_{k+1}} = \frac{r_{k+1}}{(\alpha_{66})_{k+1}} - \frac{r_k}{(\alpha_{66})_k} \tag{43}$$

$$-(\alpha_{11})_k\lambda_k\mathrm{e}^{-\lambda_k b_k}C_{1,k} + (\alpha_{11})_k\lambda_k C_{2,k} = -(\alpha_{11})_{k+1}\lambda_{k+1}C_{1,k+1} + (\alpha_{11})_{k+1}\lambda_{k+1}\mathrm{e}^{-b_{k+1}\lambda_{k+1}}C_{2,k+1} \tag{44}$$

where $k = 1,\ldots,K$.

There are $2\times K$ equations from which the $2\times K$ unknowns ($C_{1,k}$, $C_{2,k}$, $k = 1,\ldots,K$) can be calculated for a given $\widetilde{\vartheta}$. From the shear flow the torque and the load can be calculated (for a given $\widetilde{\vartheta}$) as

$$\widehat{T} = \oint qr\,\mathrm{d}\eta \tag{45}$$

$$t = \int_0^l \widehat{T}\,\mathrm{d}x = \int_0^l \oint qr\,\mathrm{d}\eta\,\mathrm{d}x \tag{46}$$

We emphasize that the shear flow (Eq. (41)) is the exact solution of the differential equation system, and hence, they can be used even in the case when the stiffnesses of the walls differ significantly.

When the loading conditions are not sinusoidal, we can write the Fourier series expansion of the load function. We obtain the solution of the problem by summing up the solutions of the elements of the series.

4.2. Solution by the Ritz method

In the following we derive an approximate solution of the above differential equations by the Ritz method. The potential energy of the beam is

$$\Pi = \frac{1}{2}\int_0^l \oint \left(N_\xi \epsilon_\xi^o + q\gamma_{\xi\eta}^0\right) \mathrm{d}\eta\,\mathrm{d}x - \int_0^l \vartheta t\,\mathrm{d}x \tag{47}$$

where the first term is the strain energy and the second term is the work done by the external load.

The axial force per unit length is (Eq. (33))

$$N_\xi = -\int_0^l \frac{\partial q}{\partial \eta}\mathrm{d}x \tag{48}$$

Eqs. (30), (31), (38), (48) and (47) result in

$$\Pi = \frac{1}{2}\int_0^l \oint \left(\alpha_{11}\frac{l^2}{\pi^2}\left(\frac{\partial q}{\partial \eta}\right)^2 + \alpha_{66}q^2\right) \mathrm{d}\eta\,\mathrm{d}x - \int_0^l \vartheta t\,\mathrm{d}x \tag{49}$$

The shear flow is assumed to be in the form of

$$q = \widetilde{q}(\eta)\cos\frac{\pi x}{l} \tag{50}$$

$$\widetilde{q}(\eta) = \sum_{i=1}^{2K} C_i \phi_i(\eta) \tag{51}$$

where C_k are yet unknown constants and the functions ϕ_k are illustrated in Fig. 9 and are given below

$$\phi_j = \left\{\begin{array}{ll} \dfrac{\eta}{b_k} + \dfrac{1}{2} & \text{when } j = k \\ -\dfrac{\eta}{b_{k+1}} + \dfrac{1}{2} & \text{when } j = k+1 \\ 0 & \text{else} \end{array}\right\} \quad \text{when } j \leqslant k \tag{52}$$

$$\phi_j = \left\{\begin{array}{ll} -\dfrac{4\eta^2}{b_k^2} + 1 & \text{when } j = K+k \\ 0 & \text{else} \end{array}\right\} \quad \text{when } j > K$$

The shear flow on the kth wall consists of three parts

$$\widetilde{q}_k = C_{k-1}\phi_{k-1} + C_k\phi_k + C_{K+k}\phi_{K+k} \tag{53}$$

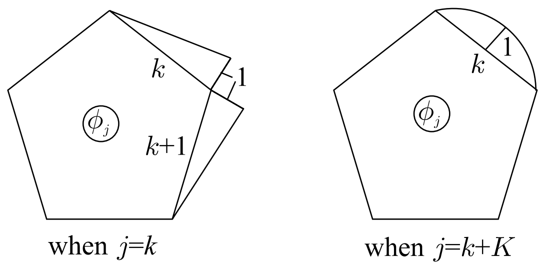

Fig. 9. Functions ϕ_j ($j = 1, 2, \ldots, 2K$, $k = 1, 2, \ldots, K$).

By substituting Eqs. (50) and (51) into Eq. (49) we obtain

$$\Pi = \frac{1}{2}\mathbf{c}^{\mathrm{T}}[F]\mathbf{c} - \mathbf{c}^{\mathrm{T}}\mathbf{f} \tag{54}$$

where the kth element of the $\mathbf{c}$ and $\mathbf{f}$ vectors are

$$\mathbf{c}_k = C_k \qquad \mathbf{f}_k = \int \vartheta \phi_k r_k \, \mathrm{d}\eta \quad k = 1, \ldots, K \tag{55}$$

and the ik element of matrix $[F]$ is

$$F_{ik} = \sum_{k=1}^{K} \int_{-b_k/2}^{b_k/2} \left((\alpha_{11})_k \frac{l^2}{\pi^2} \frac{\partial \phi_k}{\partial \eta} \frac{\partial \phi_i}{\partial \eta} + (\alpha_{66})_k \phi_k \phi_i \right) \mathrm{d}\eta \tag{56}$$

According to the principle of stationary potential energy, we have

$$\Pi = \frac{1}{2}\mathbf{c}^{\mathrm{T}}[F]\mathbf{c} - \mathbf{c}^{\mathrm{T}}\mathbf{f} = \text{stationary!} \tag{57}$$

The necessary condition for Eq. (57) is $\partial\pi/\partial C_k = 0$, which results in the following equation:

$$[F]\mathbf{c} - \mathbf{f} = 0 \tag{58}$$

The unknown constants can be calculated as

$$\mathbf{c} = [F]^{-1}\mathbf{f} \tag{59}$$

When the constants are known, q can be calculated by Eq. (50). From q the torque load can be calculated by Eq. (46).

5. Beam theory

All the six assumptions of Section 1 are valid, the last one is reiterated here.
The axial strain is (Eq. (9))

$$\epsilon_x^o = \frac{\mathrm{d}u}{\mathrm{d}x} - y\frac{\mathrm{d}\chi_y}{\mathrm{d}x} - z\frac{\mathrm{d}\chi_z}{\mathrm{d}x} - \omega\frac{\mathrm{d}\vartheta_{\mathrm{B}}}{\mathrm{d}x} \tag{60}$$

where χ_y, χ_z and ϑ_{B} are given by Eq. (10).

5.1. Governing equations in pure torsion

For convenience we separate the shear flow q as

$$q = q_0 + q_\omega \tag{61}$$

where q_0 is uniform around the circumference (Fig. 10).
These shear flows result in the following torques:

$$\widehat{T}_{\mathrm{sv}} = \oint q_0 r \, \mathrm{d}\eta \qquad \widehat{T}_\omega = \oint q_\omega r \, \mathrm{d}\eta \tag{62}$$

and the total torque is

$$\widehat{T} = \widehat{T}_{\mathrm{sv}} + \widehat{T}_\omega \tag{63}$$

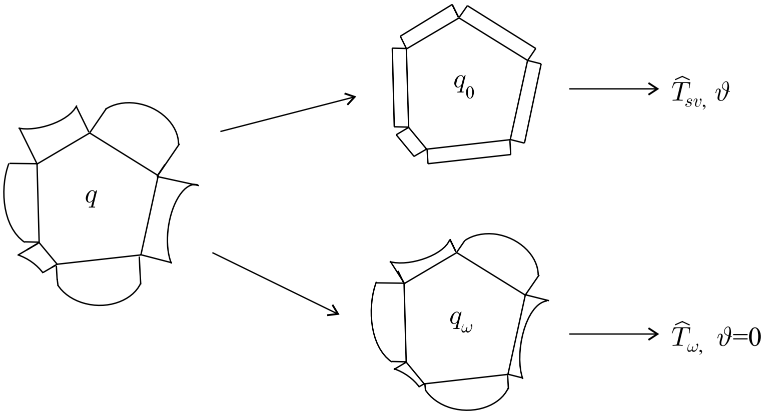

Fig. 10. Shear flow $q = q_0 + q_\omega$.

The shear flow results in a rate of twist

$$\vartheta = \frac{\oint q\alpha_{66}\,\mathrm{d}\eta}{2A} \tag{64}$$

where A is the enclosed area.

We separate q (Eq. (61)) such that q_ω does not cause a twist. Hence we have

$$\frac{\oint q_\omega\alpha_{66}\,\mathrm{d}\eta}{2A} = 0 \tag{65}$$

and

$$\vartheta = \frac{\oint q_0\alpha_{66}\,\mathrm{d}\eta}{2A} = q_0\frac{\oint \alpha_{66}\,\mathrm{d}\eta}{2A} \tag{66}$$

We define the bimoment $\widehat{M}_\omega^*$ such that the first derivative of $\widehat{M}_\omega^*$ is equal to $\widehat{T}_\omega$. (See Eq. (11) for open section beams.)

$$\widehat{T}_\omega = \frac{\mathrm{d}\widehat{M}_\omega^*}{\mathrm{d}x} \tag{67}$$

Note, however, that Vlasov's definition for the bimoment, $\widehat{M}_\omega = \oint N_\xi\omega\,\mathrm{d}\eta$, is different, Eqs. (48), (67) and (62) give

$$\widehat{M}_\omega^* = \frac{\int\oint q_\omega r\,\mathrm{d}\eta\,\mathrm{d}x}{-\oint\int\frac{\mathrm{d}q}{\mathrm{d}\eta}\mathrm{d}x\,\omega\,\mathrm{d}\eta}\widehat{M}_\omega \tag{68}$$

With Eq. (67) we obtain the same equilibrium equations as for open section beams

$$\begin{bmatrix} -\frac{\mathrm{d}}{\mathrm{d}x} & -\frac{\mathrm{d}}{\mathrm{d}x} & \\ & -1 & \frac{\mathrm{d}}{\mathrm{d}x} \end{bmatrix} \begin{Bmatrix} \widehat{T}_{\mathrm{sv}} \\ \widehat{T}_\omega \\ \widehat{M}_\omega^* \end{Bmatrix} = \begin{Bmatrix} t \\ 0 \end{Bmatrix} \tag{69}$$

Similarly, as for open section beams, we assume that the rate of twist consists of two terms, $\vartheta = \vartheta_S + \vartheta_B$ (Eq. (6)) and write the strain–displacement relationship as (Eq. (18))

$$\begin{Bmatrix} \vartheta \\ \vartheta_S \\ \Gamma \end{Bmatrix} = \begin{bmatrix} \frac{d}{dx} & \\ \frac{d}{dx} & -1 \\ & -\frac{d}{dx} \end{bmatrix} \begin{Bmatrix} \psi \\ \vartheta_B \end{Bmatrix} \tag{70}$$

and assume that these generalized strains are related to the internal forces by

$$\begin{Bmatrix} \widehat{T}_{sv} \\ \widehat{T}_{\omega} \\ \widehat{M}^*_{\omega} \end{Bmatrix} = \begin{bmatrix} \widehat{G}I_t & & \\ & S_{\omega\omega} & \\ & & \widehat{EI}_{\omega} \end{bmatrix} \begin{Bmatrix} \vartheta \\ \vartheta_S \\ \Gamma \end{Bmatrix} \tag{71}$$

where $\widehat{G}I_t$, $S_{\omega\omega}$ and $\widehat{EI}_{\omega}$ are yet unknown stiffnesses. In the following section we will determine expressions for the stiffnesses to obtain an acceptable description for the beam with the above governing equations.

5.2. Replacement stiffnesses in pure torsion

To determine the stiffnesses $\widehat{G}I_t$, $S_{\omega\omega}$ and $\widehat{EI}_{\omega}$ of the beam, we will make use of the derived solution for the case of beams subjected to a *sinusoidal load* (Sections 4.1, 4.2, Fig. 8).

The strain energy of the beam is

$$U = \frac{1}{2}\int_0^l \left(\int \left(\underbrace{\alpha_{11} N_\xi}_{\epsilon^o_\xi} N_\xi + \underbrace{\alpha_{66} q}_{\gamma^0_{\xi\eta}} q \right) d\eta \right) dx \tag{72}$$

We introduced the internal forces, generalized strains and the stiffnesses of the beam in the previous section. By using these definitions the strain energy can be written as

$$U = \frac{1}{2}\int_0^l \left(\widehat{T}_{sv}\vartheta + \widehat{T}_{\omega}\vartheta_s + \widehat{M}^*_{\omega}\Gamma \right) dx = \frac{1}{2}\int_0^l \left(\frac{\widehat{T}^2_{sv}}{\widehat{G}I_t} + \frac{\widehat{T}^2_{\omega}}{S_{\omega\omega}} + \frac{\widehat{M}^{*2}_{\omega}}{\widehat{EI}_{\omega}} \right) dx \tag{73}$$

We recall (Eq. (38)) that for a sinusoidal load q and ϑ are trigonometrical functions, and hence, $\widehat{T}_{sv}$, $\widehat{T}_{\omega}$ and $\widehat{M}^*_{\omega}$ are also trigonometrical functions and the integration with respect to x can be performed. From Eqs. (72) and (73), together with Eq. (48) we obtain

$$U = \frac{1}{2}\frac{l}{\pi} \oint \alpha_{11} \frac{l^2}{\pi^2} \left(\frac{\partial q}{\partial \eta} \right)^2 + \alpha_{66} q^2 \, d\eta \tag{74}$$

and

$$U = \frac{1}{2}\frac{l}{\pi} \left(\frac{\widehat{T}^2_{sv}}{\widehat{G}I_t} + \frac{\widehat{T}^2_{\omega}}{S_{\omega\omega}} + \frac{\widehat{M}^{*2}_{\omega}}{\widehat{EI}_{\omega}} \right) \tag{75}$$

We introduce $q = q_0 + q_\omega$ (Eq. (61)) into Eq. (74) and obtain

$$U = \frac{1}{2}\frac{l}{\pi} \left(\int \alpha_{66} q_0^2 \, d\eta + \int \alpha_{66} q_\omega^2 \, d\eta + \frac{l^2}{\pi^2} \int \alpha_{11} \left(\frac{\partial q}{\partial \eta} \right)^2 d\eta + 2q_0 \underbrace{\int q_\omega \alpha_{66} \, d\eta}_{0} \right) \tag{76}$$

As a consequence of Eq. (65) the last term in Eq. (76) is zero.

Introducing Eqs. (62) and (67) into Eq. (75) we obtain

$$U = \frac{1}{2}\frac{l}{\pi}\left(\frac{(\oint q_0 r\,\mathrm{d}s)^2}{\widehat{GI}_t} + \frac{(\oint q_\omega r\,\mathrm{d}s)^2}{S_{\omega\omega}} + \frac{(\frac{l}{\pi}\oint q_\omega r\,\mathrm{d}s)^2}{\widehat{EI}_\omega}\right) \tag{77}$$

By comparing Eqs. (76) and (77) we have

$$\widehat{GI}_t = \frac{(\oint q_0 r\,\mathrm{d}s)^2}{\int \alpha_{66} q_0^2\,\mathrm{d}\eta} \tag{78}$$

$$S_{\omega\omega} = \frac{(\oint q_\omega r\,\mathrm{d}s)^2}{\int \alpha_{66} q_\omega^2\,\mathrm{d}\eta} \tag{79}$$

$$\widehat{EI}_\omega = \frac{(\oint q_\omega r\,\mathrm{d}s)^2}{\int \alpha_{11}(\frac{\partial^2 q}{\partial \eta^2})^2\,\mathrm{d}\eta} \tag{80}$$

q_0 is uniform around the circumference, hence Eq. (78) becomes

$$\widehat{GI}_t = \frac{(\oint r\,\mathrm{d}s)^2}{\int \alpha_{66}\,\mathrm{d}\eta} = \frac{4A^2}{\int \alpha_{66}\,\mathrm{d}\eta} \tag{81}$$

To determine $S_{\omega\omega}$ and $\widehat{EI}_\omega$ the distribution of q_ω must be known. In Sections 4.1 and 4.2 we determined q_ω, and obtained that q_ω depends on the length l, and as a consequence, $S_{\omega\omega}$ and $\widehat{EI}_\omega$ also depend on l. To derive stiffnesses which are independent of the length l we will assume that $l/b_k \gg 1$, where b_k is the width of the kth wall segment.

To calculate the stiffnesses we may either use the "exact" solution (see Eq. (41)) or the approximate solution obtained via the Ritz method (see Eq. (50)). To obtain simpler results the approximate solution will be used. First the expressions for a doubly symmetrical box beam is derived then a general cross-section undergoing pure torsion will be considered.

5.2.1. Doubly symmetrical, box section beams

We consider a simply supported, doubly symmetrical box beam seen in Fig. 11.

$\widehat{GI}_t$ is given by Eq. (81), which results in

$$\widehat{GI}_t = \frac{2b_1^2 b_2^2}{X} \tag{82}$$

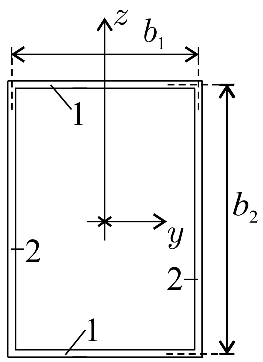

Fig. 11. Box-section beam.

where

$$X = (\alpha_{66})_1 b_1 + (\alpha_{66})_2 b_2 \tag{83}$$

The beam is subjected to a torque load $t = \tilde{t}\sin \pi x/l$ (Fig. 8). Under the applied load the beam undergoes a rate of twist $\vartheta = \tilde{\vartheta}\cos \pi x/l$, where $\tilde{\vartheta}$ is a yet unknown constant.

The box beam has four wall segments, hence the number of functions in the Ritz method is $2K = 2 \times 4 = 8$, and $\tilde{q}$ is

$$\tilde{q} = \sum_{k=1}^{8} C_k \phi_k \tag{84}$$

Because of symmetry

$$C_1 = C_2 = C_3 = C_4 \qquad C_5 = C_7 \qquad C_6 = C_8 \tag{85}$$

Hence we use the shape functions given in Fig. 12.

With these simplifications Eq. (59) becomes

$$\mathbf{c} = [F]^{-1}\mathbf{f} \tag{86}$$

where

$$\mathbf{c} = \begin{Bmatrix} C_1 \\ C_5 \\ C_6 \end{Bmatrix} \qquad \mathbf{f} = \begin{Bmatrix} A/2 \\ A/3 \\ A/3 \end{Bmatrix} \tag{87}$$

$$[F] = \frac{l^2}{\pi^2}\begin{bmatrix} \frac{\pi^2}{l^2}\frac{(\alpha_{66})_1 b_1}{2} + \frac{\pi^2}{l^2}\frac{(\alpha_{66})_2 b_2}{2} & \frac{\pi^2}{l^2}\frac{(\alpha_{66})_1 b_1}{3} & \frac{\pi^2}{l^2}\frac{(\alpha_{66})_2 b_2}{3} \\ \frac{\pi^2}{l^2}\frac{(\alpha_{66})_1 b_1}{3} & \frac{16(\alpha_{11})_1}{3b_1} + \frac{\pi^2}{l^2}\frac{8(\alpha_{66})_1 b_1}{15} & \\ \frac{\pi^2}{l^2}\frac{(\alpha_{66})_2 b_2}{3} & & \frac{16(\alpha_{11})_2}{3b_2} + \frac{\pi^2}{l^2}\frac{8(\alpha_{66})_2 b_2}{15} \end{bmatrix} \tag{88}$$

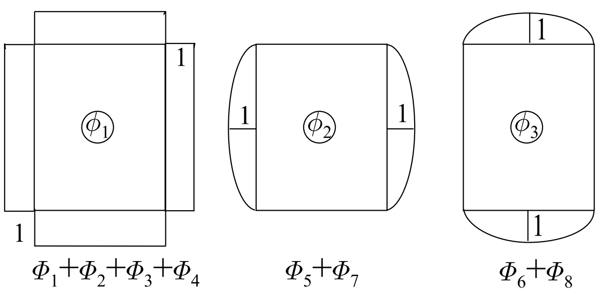

Fig. 12. Functions ϕ_k for a doubly symmetrical box beam.

Solution of Eq. (86) results in

$$C_1 = \widetilde{\vartheta}\frac{\pi^2}{l^2}A^2 \frac{\dfrac{1}{2} - \dfrac{\frac{1}{3}\frac{(\alpha_{66})_1 b_1}{3}A\frac{\pi^2}{l^2}}{A\frac{16(\alpha_{11})_1}{3b_1} + A\frac{\pi^2}{l^2}\frac{8(\alpha_{66})_1 b_1}{15}} - \dfrac{\frac{1}{3}\frac{(\alpha_{66})_2 b_2}{3}A\frac{\pi^2}{l^2}}{A\frac{16(\alpha_{11})_2}{3b_2} + A\frac{\pi^2}{l^2}\frac{8(\alpha_{66})_2 b_2}{15}}}{\dfrac{(\alpha_{66})_1 b_1}{2}A\dfrac{\pi^2}{l^2} + \dfrac{(\alpha_{66})_2 b_2}{2}A\dfrac{\pi^2}{l^2} - 2\dfrac{\left(\frac{(\alpha_{66})_1 b_1}{3}A\frac{\pi^2}{l^2}\right)^2}{A\frac{16(\alpha_{11})_1}{3b_2} + A\frac{\pi^2}{l^2}\frac{8(\alpha_{66})_1 b_1}{15}} - 2\dfrac{\left(\frac{(\alpha_{66})_2 b_2}{3}A\frac{\pi^2}{l^2}\right)^2}{A\frac{16(\alpha_{11})_2}{3b_2} + A\frac{\pi^2}{l^2}\frac{8(\alpha_{66})_{21} b_2}{15}}}$$

$$C_5 = \widetilde{\vartheta}\frac{\frac{\pi^2}{l^2}\frac{A^2}{3} - 2\frac{(\alpha_{66})_1 b_1}{3}A\frac{\pi^2}{l^2}C_1}{A\frac{16(\alpha_{11})_1}{3b_1} + A\frac{\pi^2}{l^2}\frac{8(\alpha_{66})_1 b_1}{15}} \tag{89}$$

$$C_6 = \widetilde{\vartheta}\frac{\frac{\pi^2}{l^2}\frac{A^2}{3} - 2\frac{(\alpha_{66})_2 b_2}{3}A\frac{\pi^2}{l^2}C_1}{A\frac{16(\alpha_{11})_2}{3b_2} + A\frac{\pi^2}{l^2}\frac{8(\alpha_{66})_2 b_2}{15}}$$

The Taylor series expression of these expressions with respect to $\sqrt{A}\pi/l$ are as follows:

$$C_1 = \widetilde{\vartheta}\left\{\frac{A}{X} + \frac{\pi^2}{l^2}\frac{2AY}{3X^2}\left(-\frac{(\alpha_{66})_1 b_1}{\frac{16(\alpha_{11})_1}{b_1}} + \frac{(\alpha_{66})_2 b_2}{\frac{16(\alpha_{11})_2}{b_2}}\right)\underbrace{+\frac{\pi^4}{l^4}\cdots}_{\text{neglected}}\right\}$$

$$C_5 = \widetilde{\vartheta}\left\{\frac{\pi^2}{l^2}\frac{AY}{\frac{16(\alpha_{11})_1}{b_1}X}\underbrace{+\frac{\pi^4}{l^4}\cdots}_{\text{neglected}}\right\} \tag{90}$$

$$C_6 = \widetilde{\vartheta}\left\{-\frac{\pi^2}{l^2}\frac{AY}{\frac{16(\alpha_{11})_2}{b_2}X}\underbrace{+\frac{\pi^4}{l^4}\cdots}_{\text{neglected}}\right\}$$

where

$$Y = (\alpha_{66})_2 b_2 - (\alpha_{66})_1 b_1 \tag{91}$$

and X is defined in Eq. (83). In these expressions we neglect the terms containing $(\sqrt{A}\pi/l)^i$, when $i \geqslant 4$.

The rate of twist can be calculated by Eq. (66), which is

$$\vartheta = \widetilde{\vartheta}\cos\frac{\pi x}{l} = \widetilde{q}_0\cos\frac{\pi x}{l}\frac{\oint \alpha_{66}\,\mathrm{d}\eta}{2A} \tag{92}$$

Eq. (92) gives the uniform shear flow

$$\widetilde{q}_0 = \widetilde{\vartheta}\frac{2A}{\oint \alpha_{66}\,\mathrm{d}\eta} \tag{93}$$

For the box beam $\oint \alpha_{66}\,\mathrm{d}\eta = 2(\alpha_{66})_1 b_1 + 2(\alpha_{66})_2 b_2 = 2X$ and hence

$$\widetilde{q}_0 = \widetilde{\vartheta}\frac{A}{X} \tag{94}$$

The shear flow q_ω is calculated as

$$q_\omega = q - q_0 \tag{95}$$

Eqs. (84) and (95) give

$$\widetilde{q}_\omega = (C_1 - \widetilde{q}_0)\phi_1 + C_5\phi_2 + C_6\phi_3 \tag{96}$$

By introducing Eq. (96) into (79) and (80) we obtain

$$S_{\omega\omega} = \frac{A^2}{2X^2}\frac{(\xi_2-\xi_1)^2}{\xi_1\xi_2(1+\kappa)} \qquad \widehat{EI}_\omega = \frac{A^2}{24}(\xi_2-\xi_1)^2 Z \tag{97}$$

where

$$Z = \frac{b_1}{(\alpha_{11})_1} + \frac{b_2}{(\alpha_{11})_2}$$

$$\xi_1 = \frac{b_1(\alpha_{66})_1}{X} \qquad \xi_2 = 1 - \xi_1 \tag{98}$$

$$\kappa = \frac{\xi_1\eta_1^2 + \xi_2\eta_2^2}{5\xi_1\xi_2(\eta_1+\eta_2)^2} \tag{99}$$

$$\eta_1 = \frac{b_1/(\alpha_{11})_1}{Z} \qquad \eta_2 = 1 - \eta_1 \tag{100}$$

(X is given by Eq. (83).)

5.2.2. General cross-section beams

We consider a thin-walled closed section beam consisting of K plane wall segments (Fig. 3). The torsional stiffness $\widehat{GI}_t$ is given by Eq. (81) which results in

$$\widehat{GI}_t = \frac{4A^2}{\sum_{k=1}^{K}(\alpha_{66})_k b_k} \tag{101}$$

where b_k and $(\alpha_{66})_k$ are the width and the shear compliance of the kth wall segment, and A is the enclosed area.

The beam is subjected to a torque load $t = \widetilde{t}\sin\pi x/l$ and the beam undergoes a rate of twist $\vartheta = \widetilde{\vartheta}\cos\pi x/l$. The shear flow of the beam is approximated by (see Eq. (50))

$$\widetilde{q} = \sum_{k=1}^{2K} C_k\phi_k \tag{102}$$

where ϕ_k is illustrated in Fig. 9, and C_k are yet unknown constants. The equation to determine these constants were derived in Section 4.2, and is reiterated below

$$[F]\mathbf{c} = \mathbf{f} \tag{103}$$

where

$$\mathbf{c} = \begin{Bmatrix} C_1 \\ C_2 \\ \vdots \\ C_{2K} \end{Bmatrix} \qquad \mathbf{f} = \begin{Bmatrix} f_1 \\ f_2 \\ \vdots \\ f_{2K} \end{Bmatrix} \tag{104}$$

and $[F]$ is

$$[F] = \underbrace{\frac{l^2}{\pi^2}\begin{bmatrix}[A_1] & \\ & [A_2]\end{bmatrix}}_{[A]} + \underbrace{\begin{bmatrix}[B_1] & [B_2] \\ [B_2]^{\mathrm{T}} & [B_3]\end{bmatrix}}_{[B]} \tag{105}$$

where the elements of vector $\mathbf{f}$ and matrices $[A_1]$, $[A_2]$, $[B_1]$, $[B_2]$ and $[B_3]$ are given in Table 2.

Solution of Eq. (103) is assumed to be in the form of

$$\mathbf{c} = \widetilde{\mathbf{c}} + \frac{\pi^2}{l^2}\widetilde{\widetilde{\mathbf{c}}} + \frac{\pi^4}{l^4}\widetilde{\widetilde{\widetilde{\mathbf{c}}}} + \cdots \tag{106}$$

By introducing Eqs. (105) and (106) into Eq. (103) we obtain

$$\left(\frac{l^2}{\pi^2}[A] + [B]\right)\left(\widetilde{\mathbf{c}} + \frac{\pi^2}{l^2}\widetilde{\widetilde{\mathbf{c}}} + \frac{\pi^4}{l^4}\widetilde{\widetilde{\widetilde{\mathbf{c}}}} + \cdots\right) = \mathbf{f} \tag{107}$$

which gives

$$[A]\widetilde{\mathbf{c}} + \frac{\pi^2}{l^2}([B]\widetilde{\mathbf{c}} + [A]\widetilde{\widetilde{\mathbf{c}}}) + \underbrace{\frac{\pi^4}{l^4}\cdots}_{\text{neglected}} = \frac{\pi^2}{l^2}\mathbf{f} \tag{108}$$

In this equation we neglect the terms π^i/l^i when $i \geqslant 4$. In order to obtain the "best" solution we make equal the multipliers of π^i/l^i in the two sides of Eq. (108), and write

$$[A]\widetilde{\mathbf{c}} = 0 \tag{109}$$

$$[B]\widetilde{\mathbf{c}} + [A]\widetilde{\widetilde{\mathbf{c}}} = \mathbf{f} \tag{110}$$

Matrix $[A]$ is singular. The non-trivial solution of (Eq. (109)) is

$$\widetilde{\mathbf{c}}_1 = \widetilde{\mathbf{c}}_2 = \cdots = \widetilde{\mathbf{c}}_K = \text{const} \qquad \widetilde{\mathbf{c}}_{K+1} = \widetilde{\mathbf{c}}_{K+2} = \cdots = \widetilde{\mathbf{c}}_{2K} = 0 \tag{111}$$

The choice of the constant is not unambiguous. Here we propose the constant value to be equal to the shear flow resulting in an infinitely long beam. Hence we write (Eq. (93)):

Table 2
Elements of matrices $[A_1]$, $[A_2]$, $[B_1]$, $[B_2]$, $[B_3]$ and vector $\mathbf{f}$

$[A_1]$	$\begin{cases}\frac{(\alpha_{11})_j}{b_j} + \frac{(\alpha_{11})_i}{b_i} & i = j \\ -\frac{(\alpha_{11})_i}{b_i} & \text{when } i = j+1 \\ -\frac{(\alpha_{11})_j}{b_j} & i = j-1 \\ 0 & \text{else}\end{cases}$	$[B_1]$	$\begin{cases}\frac{(\alpha_{66})_i b_i}{3} + \frac{(\alpha_{66})_j b_j}{3} & i = j \\ \frac{(\alpha_{66})_i b_i}{6} & \text{when } i = j+1 \\ \frac{(\alpha_{66})_j b_j}{6} & i = j-1 \\ 0 & \text{else}\end{cases}$
$[A_2]$	$\begin{cases}\frac{16(\alpha_{11})_i}{3b_i} & \text{when } i = j \\ 0 & \text{else}\end{cases}$	$[B_2]$	$\begin{cases}\frac{(\alpha_{66})_i b_i}{3} & i = j \\ \frac{(\alpha_{66})_j b_j}{3} & \text{when } i = j-1 \\ 0 & \text{else}\end{cases}$
$\mathbf{f}$	$\begin{cases}\frac{b_k r_k}{2} & \text{when } k \leqslant K \\ \frac{2b_k r_k}{3} & k \geqslant K\end{cases}$	$[B_3]$	$\begin{cases}\frac{8(\alpha_{66})_i b_i}{15} & \text{when } i = j \\ 0 & \text{else}\end{cases}$

$$\widetilde{\mathbf{c}}_1 = \widetilde{\mathbf{c}}_2 = \cdots = \widetilde{\mathbf{c}}_K = \widetilde{q}_0 = \widetilde{\vartheta}\frac{2A}{\sum_{k=1}^{K}(\alpha_{66})_k b_k} \tag{112}$$

Eq. (110) gives $2K$ equations to determine $\widetilde{\widetilde{\mathbf{c}}}$.

$$[A]\widetilde{\widetilde{\mathbf{c}}} = \mathbf{f} - [B]\widetilde{\mathbf{c}} \tag{113}$$

$[A]$ is singular and, consequently, the elements of $\widetilde{\widetilde{\mathbf{c}}}$ cannot be determined unambiguously from Eq. (113) only. However we have an additional condition which is discussed below.

We may observe (see Eqs. (111) and (112)) that

$$\widetilde{q}_0 = \sum_{k=1}^{2K}\widetilde{C}_k\phi_k = \sum_{k=1}^{K}\widetilde{C}_k\phi_k \tag{114}$$

and, consequently (see Eqs. (61), (101) and (106))

$$\widetilde{q}_\omega = \frac{\pi^2}{l^2}\sum_{k=1}^{2K}\widetilde{\widetilde{C}}_k\phi_k \tag{115}$$

We now make use of Eq. (65), which can be given in the following form:

$$\sum_{k=1}^{K}\widetilde{\widetilde{C}}_k\left(\frac{(\alpha_{66})_k b_k}{2} + \frac{(\alpha_{66})_{k+1} b_{k+1}}{2}\right) + \sum_{k=1}^{K}\widetilde{\widetilde{C}}_{k+K}\frac{2(\alpha_{66})_k b_k}{3} = 0 \tag{116}$$

The elements of $\widetilde{\widetilde{C}}_k$ are determined from the following $2K$ equations: The 2nd through $2K$th equations of Eq. (113)

$$\sum_{j=1}^{2K}A_{jk}\widetilde{\widetilde{C}}_k = \mathbf{f}_k - \widetilde{q}_0\sum_{j=1}^{K}B_{jk} \quad k = 2,3,\ldots,2K \tag{117}$$

and from Eq. (116). We substitute $\widetilde{\widetilde{C}}_k$ into Eq. (115) and then into Eqs. (79) and (80) which results in

$$S_{\omega\omega} = \frac{\left(\sum_{k=1}^{K} r_k b_k\left(\frac{\widetilde{\widetilde{C}}_{k-1}+\widetilde{\widetilde{C}}_k}{2} + \frac{2}{3}\widetilde{\widetilde{C}}_{K+k}\right)\right)^2}{\sum_{k=1}^{K}(\alpha_{66})_k b_k\left(\frac{\widetilde{\widetilde{C}}_{k-1}^2+\widetilde{\widetilde{C}}_k^2+\widetilde{\widetilde{C}}_{k-1}\widetilde{\widetilde{C}}_k}{3} + \frac{8}{15}\widetilde{\widetilde{C}}_{K+k}^2 + \frac{2}{3}\widetilde{\widetilde{C}}_{K+k}\left(\widetilde{\widetilde{C}}_{k-1} + \widetilde{\widetilde{C}}_k\right)\right)}$$

$$\widehat{EI}_\omega = \frac{\left(\sum_{k=1}^{K} r_k b_k\left(\frac{\widetilde{\widetilde{C}}_{k-1}+\widetilde{\widetilde{C}}_k}{2} + \frac{2}{3}\widetilde{\widetilde{C}}_{K+k}\right)\right)^2}{\sum_{k=1}^{K}\frac{(\alpha_{11})_k}{b_k}\left(\left(-\widetilde{\widetilde{C}}_{k-1} + \widetilde{\widetilde{C}}_k\right)^2 + \frac{16}{3}\widetilde{\widetilde{C}}_{K+k}^2\right)} \tag{118}$$

Note that we derived explicit expressions for $\widehat{GI}_t$, $S_{\omega\omega}$ and $\widehat{EI}_\omega$ which are independent of the beam's length.

5.3. Bending–torsion coupling—unsymmetrical beams

In the previous section we considered beams undergoing pure torsion.

As a rule beams undergo lateral and torsional deformations simultaneously. By combining Eq. (16) and Eq. (71) we write

$$\begin{Bmatrix} \gamma_y \\ \gamma_z \\ \vartheta \\ \vartheta_s \end{Bmatrix} = \begin{bmatrix} s_{yy} & s_{yz} & s_{y0} & s_{y\omega} \\ s_{yz} & s_{zz} & s_{z0} & s_{z\omega} \\ s_{y0} & s_{z0} & s_{00} & s_{0\omega} \\ s_{y\omega} & s_{z\omega} & s_{\omega 0} & s_{\omega\omega} \end{bmatrix} \begin{Bmatrix} \widehat{V}_y \\ \widehat{V}_z \\ \widehat{T}_{sv} \\ \widehat{T}_\omega \end{Bmatrix} \tag{119}$$

where s_{ij} are the shear compliances. To determine the shear compliances we write the shear flow as

$$q = q_y + q_z + q_T \tag{120}$$

where q_y, q_z, q_T are the shear flows from the shear forces $\widehat{V}_y$ and $\widehat{V}_z$ and from the torque $\widehat{T}$, respectively. The shear flow from the torque is separated as (Eq. (61)):

$$q_T = q_0 + q_\omega \tag{121}$$

Hence we have

$$\begin{aligned} \widehat{V}_y &= \int q_y \, \mathrm{d}\eta \\ \widehat{V}_z &= \int q_z \, \mathrm{d}\eta \\ \widehat{T}_{sv} &= \int r q_0 \, \mathrm{d}\eta \\ \widehat{T}_\omega &= \int r q_\omega \, \mathrm{d}\eta \end{aligned} \tag{122}$$

The compliances are determined similarly as for pure torsion. The strain energy of the beam is

$$U = U_N + U_q = \frac{1}{2} \int \alpha_{11} \frac{l^2}{\pi^2} \left(\frac{\partial^2 q}{\partial \eta^2} \right)^2 \mathrm{d}\eta + \frac{1}{2} \int \alpha_{66} q^2 \, \mathrm{d}\eta \tag{123}$$

With the internal forces in Eq. (119) we write

$$\begin{aligned} U_q = {} & \frac{1}{2} \widehat{V}_y^2 s_{yy} + \frac{1}{2} \widehat{V}_z^2 s_{zz} + \frac{1}{2} \widehat{T}_{sv}^2 s_{00} + \frac{1}{2} \widehat{T}_\omega^2 s_{\omega\omega} + \widehat{V}_y \widehat{V}_z s_{yz} + \widehat{V}_y \widehat{T}_{sv} s_{y0} + \widehat{V}_z \widehat{T}_{sv} s_{z0} + \widehat{V}_y \widehat{T}_\omega s_{y\omega} \\ & + \widehat{V}_z \widehat{T}_\omega s_{z\omega} + \widehat{T}_{sv} \widehat{T}_\omega s_{\omega 0} \end{aligned} \tag{124}$$

By introducing Eqs. (120 and 121) into the second part of Eq. (123) we have

$$U_q = \frac{1}{2} \int \alpha_{66} \left(q_y^2 + q_z^2 + q_0^2 + q_\omega^2 + 2q_y q_z + 2q_y q_0 + 2q_y q_\omega + 2q_z q_0 + 2q_z q_\omega + 2q_0 q_\omega \right) \mathrm{d}\eta \tag{125}$$

By comparing Eq. (124) and (125) we obtain

$$s_{ij} = \frac{\int \alpha_{66} q_i q_j \, \mathrm{d}\eta}{\int q_i(r) \mathrm{d}\eta \int q_j(r) \mathrm{d}\eta} \quad i,j = y, z, 0, w \tag{126}$$

The shear flows q_0 and q_ω can be calculated according to the previous section, while q_y and q_z according to classical textbooks.

Eq. (65) results in

$$s_{\omega 0} = 0 \tag{127}$$

6. Verification

In this section we demonstrate the utility of the presented theory through numerical examples.

First we consider a simply supported beam subjected to a sinusoidal load (Fig. 8).

The cross-section of the beam is shown in Fig. 13a. The material properties are given in Table 3. The thickness of the wall is 2 mm.

For simplicity the dimensions are omitted below (the forces are given in N and the distances in mm). With these properties the value of α_{11} and α_{66} for the flanges (subscript 1) and for the webs (subscript 2) are

$$\begin{aligned} (\alpha_{11})_1 &= 3.0799 \times 10^{-5} \qquad (\alpha_{66})_1 = 1.432 \times 10^{-5} \\ (\alpha_{11})_2 &= 3.3784 \times 10^{-6} \qquad (\alpha_{66})_2 = 1.0989 \times 10^{-4} \end{aligned} \tag{128}$$

The stiffnesses of the beam are calculated by Eqs. (82), (97), (83) and (91). With $b_1 = 50$ mm and $b_2 = 70$ mm we obtain

$$\begin{aligned} &A = 3500 \qquad X = 0.0084083 \qquad Y = -0.0069763 \\ &\widehat{GI}_t = 2.9138 \times 10^9 \qquad S_{\omega\omega} = 2.1308 \times 10^9 \qquad \widehat{EI}_\omega = 7.8507 \times 10^{12} \end{aligned} \tag{129}$$

The twist is given in Appendix A (Eq. (A.11)) for $k = 1$

$$\begin{aligned} \psi &= \sum \tilde{\psi} \sin \frac{\pi x}{l} \\ \tilde{\psi} &= \frac{l^2}{\pi^2 \left(\widehat{GI}_t + S_{\omega\omega} \left(1 - \frac{S_{\omega\omega}}{S_{\omega\omega} + \widehat{EI}_\omega \frac{\pi^2}{l^2}} \right) \right)} \tilde{t} \end{aligned} \tag{130}$$

For $l = 150$ mm and $t = \tilde{t} \sin \pi x / l$, at the midspan, we have

$$\tilde{\psi} = 5.3893 \times 10^{-7} \tilde{t} \tag{131}$$

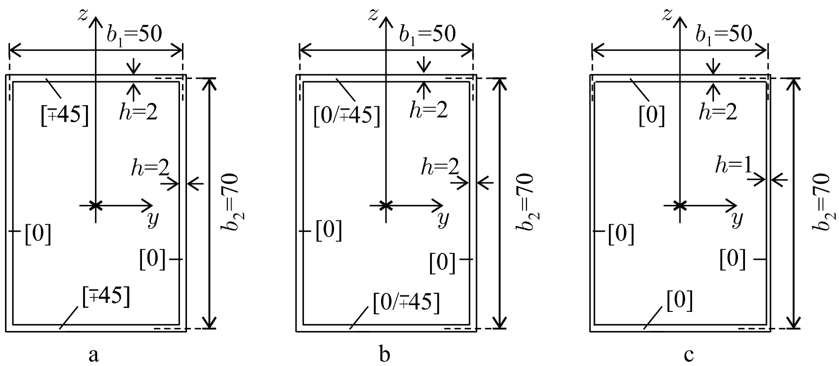

Fig. 13. The cross-sections in the numerical examples.

Table 3
Material properties of a graphite epoxy ply

	E_1 [MPa]	E_2 [MPa]	G_{12} [MPa]	ν_{12}
T300/934	148 000	9650	4550	0.3

We calculated the twist of the middle section also by solving the differential equation system of the walls (Section 4.1, "accurate solution"), and we obtained

$$\widetilde{\psi} = 5.3352 \times 10^{-7}\widetilde{t} \tag{132}$$

The difference is only −1.01%. Note that by neglecting $S_{\omega\omega}$ we obtain $\widetilde{\psi} = 3.5870 \times 10^{-7}\widetilde{t}$ and by using Kristek's theory Kristek (1979) $\widetilde{\psi} = 4.4204 \times 10^{-7}\widetilde{t}$. The inaccuracy of these values are 32.77% and 17.15%, respectively, which are not acceptable.

In Fig. 14 we show the results for the same beam as a function of the beam length.

We assumed that the maximum rate of twist on the beam is unity ($\vartheta = \cos \pi x/l$) and we calculated the torque ($\widehat{T} = \widetilde{T} \cos \pi x/l$) which results in ϑ. In this figure we included the results for the case when only $\widehat{GI}_{\mathrm{t}}$ is considered and when only $\widehat{GI}_{\mathrm{t}}$ and $\widehat{EI}_{\omega}$ are taken into account, however $S_{\omega\omega}$ is assumed to be infinity. The results of Kristek's theory are also presented. The shorter the beam the more important the effect of $S_{\omega\omega}$.

For very short beams even the presented method becomes inaccurate. (The reason is that the function of q differs very much from a second order parabola (see Eq. (84)) for very short beams. Because of the same reason, Vlasov's theory is also inaccurate for short beams.) In these cases we should model the beam as a shell structure. The question arises at which beam length may the above theory be used? By considering Eqs. (89) and (90) we can see that the term $\frac{\pi^2}{l^2}\frac{8(\alpha_{66})_k b_k}{15}$ was neglected with respect to $\frac{16(\alpha_{11})_k}{3b_k}$ for each wall segment. Hence we write

$$\frac{16(\alpha_{11})_k}{3b_k} \gg \frac{\pi^2}{l^2}\frac{8(\alpha_{66})_k b_k}{15} \tag{133}$$

which yields

$$\frac{l^2}{b_k^2}\frac{(\alpha_{11})_k}{(\alpha_{66})_k} \gg 1 \tag{134}$$

We made several numerical comparisons; on the basis of these we found that our beam theory can be used when

$$\delta = \frac{l}{K}\sum_{k=1}^{K}\frac{1}{b_k}\sqrt{\frac{(\alpha_{11})_k}{(\alpha_{66})_k}} \geqslant 1\text{–}3 \tag{135}$$

where K is the number of the wall segments. (Note that for isotropic beams the above condition gives $l/K\sum_{k=1}^{K} b_k > 2\text{–}5$.)

For the beam shown in Fig. 13a and $l = 150$, $\delta = 2.3877$. In Fig. 14 the results, as a function of δ, are also presented (see top axis).

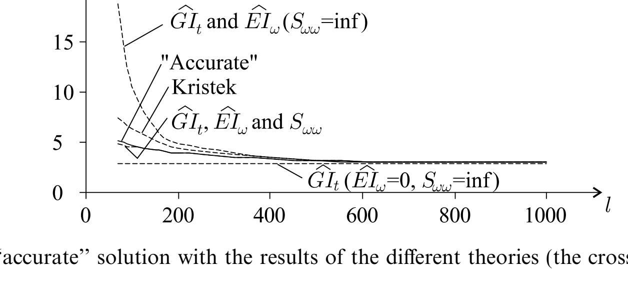

Fig. 14. Comparison of the "accurate" solution with the results of the different theories (the cross-section is given in Fig. 13a).

We also considered the cross-section shown in Fig. 13b. The compliances of the walls are

$$(\alpha_{11})_1 = 6.1000 \times 10^{-6} \qquad (\alpha_{66})_1 = 2.3498 \times 10^{-5} \tag{136}$$

$$(\alpha_{11})_2 = 3.3784 \times 10^{-6} \qquad (\alpha_{66})_2 = 1.0989 \times 10^{-4} \tag{137}$$

The results are shown in Fig. 15. In this figure we included our beam solution, the ''accurate'' solution, and Kristek's modified solution. It is seen that for beams, when $\delta \geqslant 1$, the presented beam model is acceptable.

Note that for the case when the layup and the thickness of the wall segments are identical the simple theory, considering $\widehat{GI}_t$ only, is applicable.

For further verification we considered beams with artificial materials, where the compliances differ significantly from each other. The values of α_{11} and α_{66} are shown in Fig. 16. The results of our calculations are shown in Figs. 17 and 18.

Then we consider a cantilever beam subjected to a concentrated torque at the end $T = 1280$ (Fig. 19). The length of the beam l is 1000 mm. The cross-section is shown in Fig. 13c.

The compliances of the walls are

$$(\alpha_{11})_1 = 1.6892 \times 10^{-6} \qquad (\alpha_{66})_1 = 5.4945 \times 10^{-5} \tag{138}$$

$$(\alpha_{11})_2 = 6.7568 \times 10^{-6} \qquad (\alpha_{66})_2 = 2.1978 \times 10^{-4} \tag{139}$$

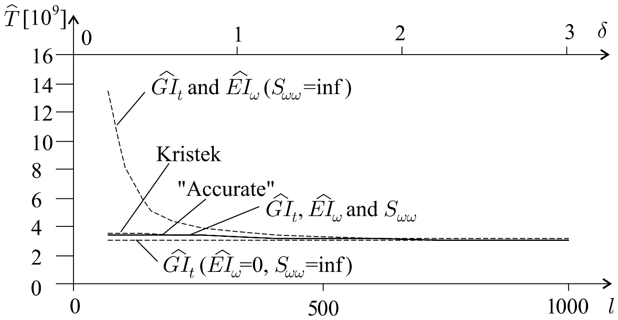

Fig. 15. Comparison of the ''accurate'' solution with the results of the different theories (the cross-section is given in Fig. 13b).

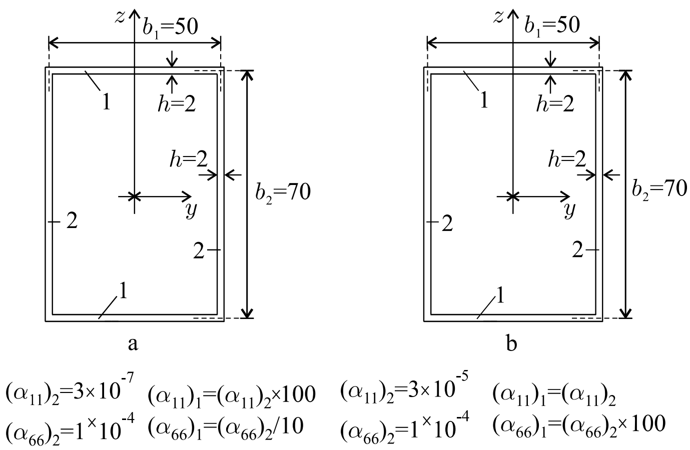

Fig. 16. Cross-sections with artificial materials.

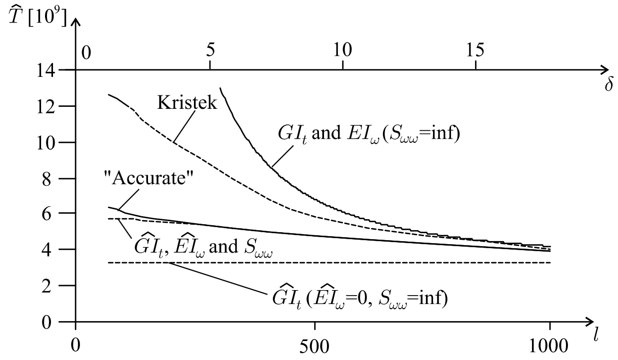

Fig. 17. Comparison of the "accurate" solution with the results of the different theories (the cross-section is given in Fig. 16a).

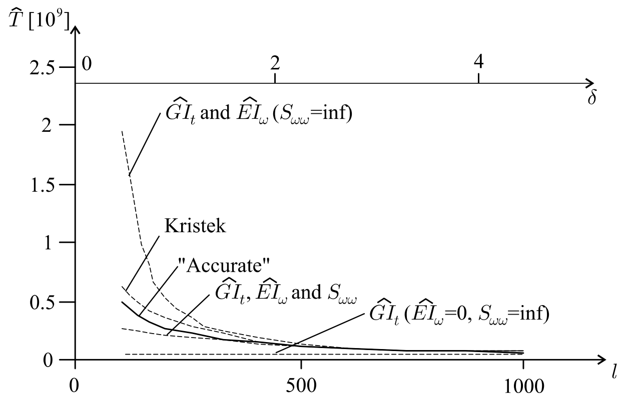

Fig. 18. Comparison of the "accurate" solution with the results of the different theories (the cross-section is given in Fig. 16b).

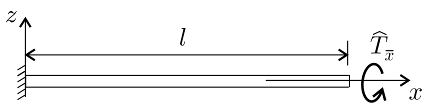

Fig. 19. Cantilever beam subjected to a torque at the end.

The stiffnesses of the beam $\widehat{GI}_t$, $S_{\omega\omega}$ and $\widehat{EI}_\omega$ are given by Eqs. (82) and (97):

$$\widehat{GI}_t = 1.3512 \times 10^9 \qquad S_{\omega\omega} = 1.0479 \times 10^8 \qquad \widehat{EI}_\omega = 9.9078 \times 10^{12} \tag{140}$$

The function of the twist is given by Eq. (B.7) in Appendix B:

$$\psi = C_1 + C_2 x + C_3 e^{\lambda(x-L)} + C_4 e^{-\lambda x} \tag{141}$$

where $\lambda = 0.0077$, $C_1 = -0.5361 \times 10^{-4}$, $C_2 = 0.0095 \times 10^{-4}$, $C_3 = 0$ and $C_4 = 0.5361 \times 10^{-4}$.

The rate of twist (i.e. the first derivative of the twist) is

$$\vartheta = C_2 + C_3 \lambda e^{\lambda(x-L)} - C_4 \lambda e^{-\lambda x} = 10^{-4} \times (0.0095 - 0.0041 \times e^{-0.0077x}) \tag{142}$$

The rate of twist was also calculated by the ANSYS FE program. The results are compared to each other in Fig. 20. It can be seen that the analytical and numerical calculations agree well.

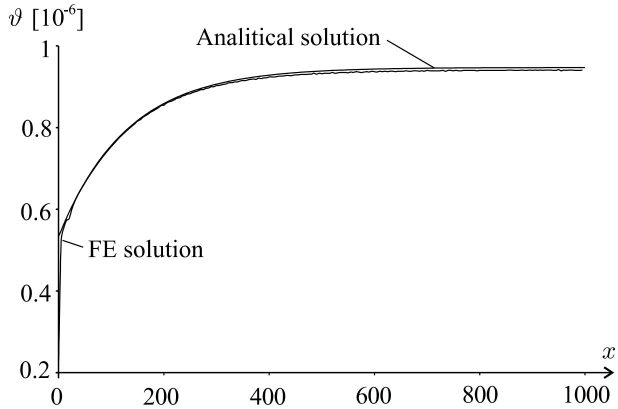

Fig. 20. Rate of twist of a cantilever beam subjected to a torque at the end (the cross-section is given in Fig. 16c).

7. Conclusions

We gave the governing differential equation system of thin-walled, closed section, orthotropic beams subjected to a torque load. We have solved the problem for sinusoidal load. Solution only for isotropic case can be found in the literature (Vlasov, 1961), which gives inaccurate results when the stiffnesses of the wall segments differ from each other significantly.

We presented a beam theory for thin-walled, closed section, orthotropic beams taking the restrained warping and the shear deformation into account. They play an important role when the stiffnesses of the walls (thickness and/or the layup) are significantly different from each other.

We gave numerical examples to demonstrate the accuracy of our beam model. The restrained warping and the shear deformation may affect the torsional stiffness of the beam, and consequently, the buckling load and the vibration characteristics. The expressions presented in Kollár and Springer (2003) and in Sapkás and Kollár (2002) for the buckling load and in Kollár (2001) for the period of vibration of *open section* beams can be applied directly for closed section beams: in the presented expressions the stiffnesses $\widehat{GI}_t$, $S_{\omega\omega}$, $\widehat{EI}_\omega$ derived in this paper must be used.

We must note, however that in most of the practical cases the torsional stiffnesses of closed section beams are relatively high and the presented effect is significant only for relatively short beams. For these cases either the presented model must be used or the beam must be modeled by shell (or 3D) finite elements.

Acknowledgement

This work was supported by the Hungarian Academy of Sciences, which is highly appreciated.

Appendix A. Simply supported beam

We consider a simply supported beam with the length l subjected to a torque with arbitrary distribution. The torque load is represented by its Fourier series expansion:

$$t = \sum \tilde{t}_k \sin \frac{\pi k x}{l} \tag{A.1}$$

In the following we determine the rate of twist for the kth element of the load: $\tilde{t}_k \sin \pi kx/l$.

The governing equations are given by Eqs. (69)–(71) which—for pure torsion—are reiterated below

$$\begin{Bmatrix} \vartheta \\ \vartheta_S \\ \Gamma \end{Bmatrix} = \begin{bmatrix} \frac{d}{dx} & \\ \frac{d}{dx} & -1 \\ & -\frac{d}{dx} \end{bmatrix} \begin{Bmatrix} \psi \\ \vartheta_B \end{Bmatrix} \tag{A.2}$$

$$\begin{Bmatrix} \widehat{T}_{sv} \\ \widehat{T}_\omega \\ \widehat{M}^*_\omega \end{Bmatrix} = \begin{bmatrix} \widehat{GI}_t & & \\ & S_{\omega\omega} & \\ & & \widehat{EI}_\omega \end{bmatrix} \begin{Bmatrix} \vartheta \\ \vartheta_S \\ \Gamma \end{Bmatrix} \tag{A.3}$$

$$\begin{bmatrix} -\frac{d}{dx} & -\frac{d}{dx} & \\ & -1 & \frac{d}{dx} \end{bmatrix} \begin{Bmatrix} \widehat{T}_{sv} \\ \widehat{T}_\omega \\ \widehat{M}^*_\omega \end{Bmatrix} = \begin{Bmatrix} \tilde{t}_k \sin \frac{\pi kx}{l} \\ 0 \end{Bmatrix} \tag{A.4}$$

The boundary conditions are

$$\begin{aligned} &\psi(0) = 0 \qquad \frac{d^2\psi}{dx^2}(0) = 0 \\ &\psi(l) = 0 \qquad \frac{d^2\psi}{dx^2}(l) = 0 \end{aligned} \tag{A.5}$$

The rate of twist ψ, and ϑ_B are assumed in the form

$$\psi_k = \tilde{\psi}_k \sin \frac{\pi kx}{l} \tag{A.6}$$

$$\vartheta_{Bk} = \tilde{\vartheta}_{Bk} \cos \frac{\pi kx}{l} \tag{A.7}$$

which satisfy the boundary conditions (Eq. (A.5)).

By introducing ψ_k and ϑ_{Bk} into Eqs. (A.2)–(A.4) we obtain

$$\begin{Bmatrix} \vartheta \\ \vartheta_S \\ \Gamma \end{Bmatrix} = \begin{bmatrix} \frac{d}{dx} & \\ \frac{d}{dx} & -1 \\ & -\frac{d}{dx} \end{bmatrix} \begin{Bmatrix} \tilde{\psi}_k \sin \frac{\pi kx}{l} \\ \tilde{\vartheta}_{Bk} \cos \frac{\pi kx}{l} \end{Bmatrix} = \begin{Bmatrix} \tilde{\psi}_k \frac{\pi k}{l} \cos \frac{\pi kx}{l} \\ \left(\tilde{\psi}_k \frac{\pi k}{l} - \tilde{\vartheta}_{Bk} \right) \cos \frac{\pi kx}{l} \\ \tilde{\vartheta}_{Bk} \frac{\pi k}{l} \sin \frac{\pi kx}{l} \end{Bmatrix} \tag{A.8}$$

$$\begin{Bmatrix} \widehat{T}_{sv} \\ \widehat{T}_\omega \\ \widehat{M}^*_\omega \end{Bmatrix} = \begin{bmatrix} \widehat{GI}_t & & \\ & S_{\omega\omega} & \\ & & \widehat{EI}_\omega \end{bmatrix} \begin{Bmatrix} \vartheta \\ \vartheta_S \\ \Gamma \end{Bmatrix} = \begin{Bmatrix} \widehat{GI}_t \tilde{\psi}_k \frac{\pi k}{l} \cos \frac{\pi kx}{l} \\ S_{\omega\omega} \left(\tilde{\psi}_k \frac{\pi k}{l} - \tilde{\vartheta}_{Bk} \right) \cos \frac{\pi kx}{l} \\ \widehat{EI}_\omega \tilde{\vartheta}_{Bk} \frac{\pi k}{l} \sin \frac{\pi kx}{l} \end{Bmatrix} \tag{A.9}$$

$$\begin{bmatrix} -\frac{\mathrm{d}}{\mathrm{d}x} & -\frac{\mathrm{d}}{\mathrm{d}x} & \\ & -1 & \frac{\mathrm{d}}{\mathrm{d}x} \end{bmatrix} \begin{Bmatrix} \widehat{T}_{\mathrm{sv}} \\ \widehat{T}_{\omega} \\ \widehat{M}^{*}_{\omega} \end{Bmatrix} = \begin{Bmatrix} \widehat{GI}_{\mathrm{t}} \widetilde{\psi}_k \frac{\pi^2 k^2}{l^2} \sin\frac{\pi k x}{l} + S_{\omega\omega}\left(\widetilde{\psi}_k \frac{\pi^2 k^2}{l^2} - \widetilde{\vartheta}_{\mathrm{B}k} \frac{\pi k}{l} \right) \sin\frac{\pi k x}{l} \\ -S_{\omega\omega}\left(\widetilde{\psi}_k \frac{\pi k}{l} - \widetilde{\vartheta}_{\mathrm{B}k} \right) \cos\frac{\pi k x}{l} + \widehat{EI}_{\omega} \widetilde{\vartheta}_{\mathrm{B}k} \frac{\pi^2 k^2}{l^2} \cos\frac{\pi k x}{l} \end{Bmatrix} = \begin{Bmatrix} \widetilde{t}_k \sin\frac{\pi k x}{l} \\ 0 \end{Bmatrix} \tag{A.10}$$

From these equations, after algebraic manipulation, we obtain

$$\psi = \sum \widetilde{\psi}_k \sin\frac{\pi k x}{l} \tag{A.11}$$

$$\widetilde{\psi}_k = \frac{l^2}{\pi^2 k^2 \left(\widehat{GI}_{\mathrm{t}} + S_{\omega\omega} \left(1 - \frac{S_{\omega\omega}}{S_{\omega\omega} + \widehat{EI}_{\omega} \frac{\pi^2 k^2}{l^2}} \right) \right)} \widetilde{t}_k \tag{A.12}$$

The rate of twist ϑ is the derivative of ψ, which is

$$\vartheta = \sum \widetilde{\vartheta}_k \cos\frac{\pi k x}{l} \tag{A.13}$$

$$\widetilde{\vartheta}_k = \frac{l}{\pi k \left(\widehat{GI}_{\mathrm{t}} + S_{\omega\omega} \left(1 - \frac{S_{\omega\omega}}{S_{\omega\omega} + \widehat{EI}_{\omega} \frac{\pi^2 k^2}{l^2}} \right) \right)} \widetilde{t}_k \tag{A.14}$$

Appendix B. Cantilever beam subjected to a concentrated torque at the end

We consider a cantilever beam subjected to a torque $\widehat{T}$ at the end (Fig. 19).
The governing equations are given by Eqs. (69)–(71), with $t = 0$.
Substituting Eq. (70) into Eq. (71) and then into Eq. (69) we obtain

$$\begin{bmatrix} \left(\widehat{GI}_{\mathrm{t}} + S_{\omega\omega} \right) \frac{\mathrm{d}^2}{\mathrm{d}x^2} & -S_{\omega\omega} \frac{\mathrm{d}}{\mathrm{d}x} \\ -S_{\omega\omega} \frac{\mathrm{d}}{\mathrm{d}x} & S_{\omega\omega} - \widehat{EI}_{\omega} \frac{\mathrm{d}^2}{\mathrm{d}x^2} \end{bmatrix} \begin{Bmatrix} \psi \\ \vartheta_{\mathrm{B}} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{B.1}$$

The boundary conditions are

$$\begin{aligned} &\psi(0) = 0 \qquad \vartheta_{\mathrm{B}}(0) = 0 \\ &\widehat{M}^{*}_{\omega}(l) = 0 \qquad \widehat{T}(l) = \widehat{T} \end{aligned} \tag{B.2}$$

Using Eqs. (70), (71) and (3) we obtain

$$\begin{aligned} &\psi(0) = 0 \qquad \vartheta_{\mathrm{B}}(0) = 0 \\ &\frac{\mathrm{d}\vartheta_{\mathrm{B}}}{\mathrm{d}x}(l) = 0 \qquad \widehat{GI}_{\mathrm{t}} \frac{\mathrm{d}\psi}{\mathrm{d}x}(l) + S_{\omega\omega}\left(\frac{\mathrm{d}\psi}{\mathrm{d}x}(l) - \vartheta_{\mathrm{B}}(l) \right) = \widehat{T} \end{aligned} \tag{B.3}$$

We assume the solution of Eq. (B.1) as follows:

$$\begin{Bmatrix} \psi \\ \vartheta_B \end{Bmatrix} = \begin{Bmatrix} \psi_0 \\ \vartheta_{B0} \end{Bmatrix} e^{\lambda x} \tag{B.4}$$

Substituting Eq. (B.4) into Eq. (B.1), and omitting $e^{\lambda x}$, we have

$$\begin{bmatrix} \left(\widehat{GI}_t + S_{\omega\omega}\right)\lambda^2 & -S_{\omega\omega}\lambda \\ -S_{\omega\omega}\lambda & S_{\omega\omega} - \widehat{EI}_\omega \lambda^2 \end{bmatrix} \begin{Bmatrix} \psi_0 \\ \vartheta_{B0} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{B.5}$$

We obtain a non-trivial solution of Eq. (B.5) if the determinant of the matrix on the left hand side is equal to zero. This condition results in four λs:

$$\lambda_1 = \lambda_2 = 0 \qquad \lambda_3 = \sqrt{\frac{\widehat{GI}_t}{\widehat{EI}_\omega \left(1 + \frac{\widehat{GI}_t}{S_{\omega\omega}}\right)}} = \lambda \qquad \lambda_4 = -\sqrt{\frac{\widehat{GI}_t}{\widehat{EI}_\omega \left(1 + \frac{\widehat{GI}_t}{S_{\omega\omega}}\right)}} = -\lambda \tag{B.6}$$

We obtain the relationship of ψ_0 and ϑ_{B0} from the second row of Eq. (B.5)

$$\vartheta_{B0} = \psi_0 \frac{S_{\omega\omega}\lambda}{S_{\omega\omega} - \widehat{EI}_\omega \lambda^2}$$

Because of the two zero values of λ the complete solution of Eq. (B.1) is (Kreyszig, 1993)

$$\psi = C_1 + C_2 x + C_3 e^{\lambda(x-L)} + C_4 e^{-\lambda x} \tag{B.7}$$

and the expression of ϑ_B is

$$\vartheta_B = C_2 + C_3 \frac{S_{\omega\omega}\lambda}{S_{\omega\omega} - \widehat{EI}_\omega \lambda^2} e^{\lambda(x-L)} - C_4 \frac{S_{\omega\omega}\lambda}{S_{\omega\omega} - \widehat{EI}_\omega \lambda^2} e^{-\lambda x} \tag{B.8}$$

Substituting these into the expressions of the boundary conditions (Eq. (B.3)) we obtain an algebraic equation system which yields the yet unknown constants C_i $(i = 1, 2, \ldots, 4)$.

When $S_{\omega\omega}$ is large ($S_{\omega\omega} > 10 l^2 \widehat{EI}_\omega$) the results are very close to those given by the expression derived on the basis of $\widehat{GI}_t$ and $\widehat{EI}_\omega$ (while $S_{\omega\omega} = \infty$) (Megson, 1990).

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